Assignment # 2	Due on October 30th in class

1. In a metric space (X, d), prove that any open ball is an open set, and any closed ball is a closed set.

2. In a metric space (X, d), for any $M \subset X$, prove that $\mathrm{Int}(M)$ is an open set.

Hint: For any $x \in \mathrm{Int}(M)$, from the definition of interiors, there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset M$. Based on this, prove the following first: For any $y \in B(x; \varepsilon/3)$, $B(y; \varepsilon/3) \subset B(x; \varepsilon)$.

3. In a metric space (X, d), use $\mathcal{T}$ to denote the collection of all the open sets. Prove that we have the following properties for $\mathcal{T}$:

i) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.

ii) Let $\mathscr{A}$ be an index set, and assume $S_i \in \mathcal{T}$ for all $i \in \mathscr{A}$. Then $\bigcup_{i \in \mathscr{A}} S_i \in \mathcal{T}$.

iii) Let $K_1, \cdots, K_n$ be in $\mathcal{T}$. Then $\bigcap_{i=1}^{n} K_i \in \mathcal{T}$.

Remark: These three properties above give an abstract characterization of "open sets" in topological spaces.

1.

Proof. Let D be an open ball, say $D = B(x; r)$ for certain $x \in X$ and $r > 0$. We will show that D is open. In fact, for each $y \in D$, just note that

$$B(y; r - d(y, x)) \subset B(x; r) = D,$$

we are done.

Let K be a closed ball. We can assume that

$$K = \{x \in K : d(x, a) \leq r\}$$

for certain $a \in X$ and $r > 0$.

To show that D is closed, we just need to show that $X - D$ is open. In fact, for any $y \notin D$, we have $d(y, a) > r$. It then follows that

$$B(y, d(y, a) - r) \subset X - D,$$

which finishes the proof.

□

2. Just follow the hint, and it should be straightforward (uing the triangle inequality of distance structure).

3.

Proof. i) From the definiton of open sets, $\varnothing$ is open (why?).

For any $x \in X$, we always have $B(x; 1) \subset X$. Thus X is open.

ii) If $x \in \bigcap_{i \in \mathscr{A}} S_i$, then $x \in S_n$ for certain $n \in \mathscr{A}$. As S_n is open, there exists $r > 0$, such that $B(x; r) \subset S_n$. Thus $B(x; r) \subset \bigcup_{i \in \mathscr{A}} S_i$, which indicates that $\bigcup_{i \in \mathscr{A}} S_i$ is open.

iii) For $x \in \bigcap_{i=1}^n K_i$, we have $x \in K_i$ for all $1 \leq i \leq n$. As each K_i is open, there exists r_i such that

$$B(x; r_i) \subset K_i, \quad \forall\, 1 \leq i \leq n.$$

Take $r = \min\{r_1, \cdots, r_n\}$, then

$$B(x; r) \subset K_i, \quad \forall\, 1 \leq i \leq n,$$

which indicates that

$$B(x; r) \ \subset \ \bigcap_{i=1}^{n} K_i.$$

Thus $\bigcap_{i=1}^n K_i$ is open. Done.

$\square$